Large-scale probability density function for scalar field advected by high Reynolds number turbulent flow

1997 J. Phys. A: Math. Gen. 30 L77

(http://iopscience.iop.org/0305-4470/30/5/001)

J. Phys. A: Math. Gen. **30** (1997) L77–L82. Printed in the UK PII: S0305-4470(97)79377-6

LETTER TO THE EDITOR

Large-scale probability density function for scalar field advected by high Reynolds number turbulent flow

Sergei Fedotov†

Department of Mathematical Physics, Ural State University, Jekaterinburg, 620083, Russia

Received 8 November 1996, in final form 15 January 1997

Abstract. A closed equation for the one-point probability density function (PDF) for a scalar field advected by the three-dimensional random velocity field with arbitrary many spatial/ temporal scales and infrared divergence in the limit of high Reynolds number has been derived, using the functional derivative technique and renormalization theory, together with the assumption that the turbulent velocity is a homogeneous, isotropic, Gaussian random field.

It has been shown that when the spectral parameters of a random velocity field slightly deviate from their Kolmogorov–Obukhov values, the equation for the PDF in the long-time, large-distance limit can be derived exactly and well described by a conventional diffusion theory, while the Lagrangian scaling function describing the large-scale particle displacements in turbulent flow is essentially superdiffusive. The scaling procedure in the limit of high Reynolds number allows us to completely overcome the well known closure problem associated with diffusion term.

The random advection of a scalar field by turbulent flow has attracted enormous attention in past years because of its importance, both for the practical applications involving a monitor of pollutants in the atmosphere and also for our understanding the nature of turbulence itself. The fundamental difficulty in this problem is that the velocity of fully developed turbulence involves random fluctuations over a vast range of spatial and temporal scales. Therefore any theory that intends to describe the turbulent transport phenomena must take into account the entire spectrum of scales of length. A thorough discussion of this problem, its theoretical and experimental aspects, can be found in [1], and references therein.

In recent years there has been renewed interest in the advection-diffusion problem generated by the successful applications of renormalization group methods [2–5] and interesting asymptotic behaviour of the probability density function of advected scalar fields [6–8] (see also [9–13]). Recently, Avellaneda and Majda have developed an *exact* renormalization technique for the solution of eddy diffusivity problem related to the advection by a random velocity field with long-range correlations and infrared divergence [14–17]. Although a great deal of progress has been made in this theory, here the results are restricted to a problem of the derivation of eddy diffusivity equations for the transport of a passive scalar. In view of the success of the exact renormalization procedure, it would seem natural to try to extend these results and derive an effective equation for the probability density function (PDF) for a scalar field.

It is well known that the main difficulty with the PDF approach arises because of the closure problem associated with the diffusion term [13]. It is our purpose to overcome

† E-mail address: sergei.fedotov@usu.ru

0305-4470/97/050077+06$19.50

this problem by an exact renormalization theory considering a *three-dimensional* random velocity field with arbitrary many spatial and temporal scales. Our intention is to derive a closed equation for the one-point PDF for a scalar field in the long-time, large-distance limit.

We assume that the turbulence is steady, homogeneous, and isotropic with the energy spectrum [1]

$$E(k) = c l_0^{5/3} \bar{\varepsilon}^{2/3} \frac{(l_0 k)^2}{[1+(l_0 k)^2]^{11/6}} \exp(-\eta^2 k^2) \qquad 0 < k < \infty \tag{1}$$

where $\eta = (\nu^3/\bar{\varepsilon})^{1/4}$ is the dissipation length scale, l_0 is the characteristic length for the energy containing eddies, $\bar{\varepsilon}$ is the average dissipation rate, ν is the kinematic viscosity, k is the wavenumber, and c is a dimensionless constant.

In what follows we shall measure time and space in terms of the dissipation time scale $t_\mathrm{d} = (\nu/\bar{\varepsilon})^{1/2}$ and length scale η considering all other quantities as dimensionless. Since we are interested here with the limit of high Reynolds number, the ratio of the Kolmogorov length scale η to the integral length scale l_0 is a small parameter, i.e.

$$\epsilon = \frac{\eta}{l_0} \ll 1 \qquad \epsilon = R\,\mathrm{e}^{-3/4}. \tag{2}$$

Consider a non-dimensional scalar field $\varphi(t, \boldsymbol{x})$ whose dynamical evolution is specified by the advection equation

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{v}(t, \boldsymbol{x}) \cdot \nabla \varphi = D\,\nabla^2 \varphi + \alpha(\epsilon) f(\varphi) \qquad \varphi(0, \boldsymbol{x}) = \varphi_0(\epsilon \boldsymbol{x}) \tag{3}$$

where D is the inverse Prandtl number and $\boldsymbol{v}(t, \boldsymbol{x})$ is the incompressible random velocity field, i.e. $\nabla \cdot \boldsymbol{v} = 0$. It is assumed here that the initial distribution of $\varphi(t, \boldsymbol{x})$ varies only on the integral length scale. The last term in (2) represents the chemical reaction with the growth rate $\alpha(\epsilon)$ depending on the small parameter ϵ.

In this paper we will be concerned with the behaviour of the one-point PDF in the long-time, large-distance limit. Let us define the PDF as follows,

$$p^\epsilon(t, \boldsymbol{x}, \varphi) = \langle \delta(\varphi - \varphi^\epsilon(t, \boldsymbol{x})) \rangle \tag{4}$$

where

$$\varphi^\epsilon(t, \boldsymbol{x}) = \varphi\left(\frac{t}{\lambda(\epsilon)}, \frac{\boldsymbol{x}}{\epsilon}\right)$$

and $\varphi(t, \boldsymbol{x})$ is a solution of (3) corresponding to a particular realization of the random field $\boldsymbol{v}(t, \boldsymbol{x})$. The angular brackets denote an average over an ensemble of realizations of $\boldsymbol{v}(t, \boldsymbol{x})$.

Our purpose is to find such a scaling function $\lambda(\epsilon)$ so that

$$p^0(t, \boldsymbol{x}, \varphi) = \lim_{\epsilon \to 0} p^\epsilon(t, \boldsymbol{x}, \varphi) = \lim_{\epsilon \to 0} \left\langle \delta\left(\varphi - \varphi\left(\frac{t}{\lambda(\epsilon)}, \frac{\boldsymbol{x}}{\epsilon}\right)\right)\right\rangle \tag{5}$$

obeys an effective renormalized equation.

A suitable choice of scaling function $\lambda(\epsilon)$ under the renormalization procedure (the limit (5) should be non-trivial) gives us the Lagrangian scaling law for the large-scale particle displacement $x(t)$ in turbulent flow $\lambda(x^{-1}(t)) \sim t$.

Applying the scaling transformation

$$\boldsymbol{x} \to \frac{\boldsymbol{x}}{\epsilon} \qquad t \to \frac{t}{\lambda(\epsilon)} \tag{6}$$

to (3) we obtain a Cauchy problem for $\varphi^\epsilon(t,\boldsymbol{x})$

$$\frac{\partial\varphi^\epsilon}{\partial t}+\frac{\epsilon}{\lambda}\boldsymbol{v}\left(\frac{t}{\lambda},\frac{\boldsymbol{x}}{\epsilon}\right)\nabla\varphi^\epsilon=\frac{\epsilon^2 D}{\lambda}\nabla^2\varphi^\epsilon+\frac{\alpha}{\lambda}f(\varphi^\epsilon)\qquad \varphi^\epsilon(0,\boldsymbol{x})=\varphi_0(\boldsymbol{x}). \tag{7}$$

To find an equation for $p^\epsilon(t,\boldsymbol{x},\varphi)$ we use the functional derivative technique from [18, 19]. By differentiating (4) with respect to t and using (7) one can get

$$\frac{\partial p^\epsilon}{\partial t}=-\frac{\partial}{\partial\varphi}\left(\frac{\alpha}{\lambda}f(\varphi)p^\epsilon\right)-\frac{\epsilon^2 D}{\lambda}\frac{\partial}{\partial\varphi}\langle\nabla^2\varphi^\epsilon\delta(\varphi-\varphi^\epsilon)\rangle+Q^\epsilon \tag{8}$$

where Q^ϵ is

$$Q^\epsilon=-\frac{\epsilon}{\lambda}\left\langle\boldsymbol{v}\left(\frac{t}{\lambda},\frac{\boldsymbol{x}}{\epsilon}\right)\cdot\nabla\delta(\varphi-\varphi^\epsilon)\right\rangle.$$

By assuming Gaussian statistics for the velocity field $\boldsymbol{v}(t,\boldsymbol{x})$ and using the Furutsu–Novikov formula [18, 19] we find that the correlation term Q^ϵ can be written as

$$Q^\epsilon=\frac{\epsilon}{\lambda}\iint\left\langle v_i\left(\frac{t}{\lambda},\frac{\boldsymbol{x}}{\epsilon}\right)v_j\left(\frac{\tau}{\lambda},\frac{\boldsymbol{y}}{\epsilon}\right)\right\rangle\frac{\partial^2}{\partial x_i\partial\varphi}\left\langle\frac{\delta\varphi(t/\lambda,\boldsymbol{x}/\epsilon)}{\delta v_j(\tau/\lambda,\boldsymbol{y}/\epsilon)}\delta(\varphi-\varphi^\epsilon)\right\rangle\frac{\mathrm{d}\tau\,\mathrm{d}\boldsymbol{y}}{\epsilon\lambda} \tag{9}$$

where $\delta/\delta v_i$ denotes functional derivatives and summation over repeated indices is implied.

The evolution equation (8) with (9) is exact but still in an unclosed form. To proceed further we need the explicit expressions for the velocity correlation tensor and the two-time, two-point functional derivative appearing in (9). It should be noted that the response function $\delta\varphi(t/\lambda,\boldsymbol{x}/\epsilon)/\delta v_j(\tau/\lambda,\boldsymbol{y}/\epsilon)$ involves the stochastic field $\varphi^\epsilon(t,\boldsymbol{x})$ itself; hence it is impossible in general to obtain a closed equation for p^ϵ [18, 19].

Since $\boldsymbol{v}(t,\boldsymbol{x})$ is assumed to be a Gaussian field with zero mean, its statistical characteristics are completely determined by the correlation tensor [14–17, 20, 21]

$$\langle v_i(t,\boldsymbol{x})v_j(\tau,\boldsymbol{y})\rangle=\frac{1}{\pi^2}\iint \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})-\mathrm{i}\omega(t-\tau)}E(k)\frac{\tau(k)}{1+(\tau(k)\omega)^2}k^{-2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega \tag{10}$$

where

$$E(k)=\frac{ck^2\,\mathrm{e}^{-k^2}}{(\epsilon^2+k^2)^{(11/6)-(\sigma/2)}}\qquad \tau(k)=\begin{cases}(\epsilon^2+k^2)^{-(1/3)+(z/2)} & k<1\\ k^{-2} & k>1.\end{cases} \tag{11}$$

This spectral representation with two exponents σ and z may be considered as a generalization of (1). The analogous parametrized family of incompressible velocity fields was first introduced in [14]. It follows from (10), (11) that the Kolmogorov–Obukhov turbulence corresponds to the case in which $\sigma=0$ and $z=0$. Therefore, the model (10) and (11) with the variable parameters σ and z corresponds to the intermittency corrections to the Kolmogorov–Obukhov $k^{-5/3}$ law [1]. The spectral parameter σ appearing here may be thought of as representing a deviation of the energy spectrum from the classical one in the inertial range, where, as follows from (11), $E(k)=ck^{-(5/3)+\sigma}$ as $\epsilon\ll k\ll 1$.

The parameter σ may be considered as a natural measure of spatial correlations of the random velocity field. It follows from (10) that the structural function for $\boldsymbol{v}$ is

$$\langle(\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{y},t)-\boldsymbol{v}(\boldsymbol{x},t))^2\rangle=4\int_0^\infty E(k)\left(1-\frac{\sin ky}{ky}\right)\mathrm{d}k\sim y^{(2/3)+\sigma}. \tag{12}$$

Since

$$\frac{1}{2}\langle\boldsymbol{v}^2\rangle=\int_0^\infty E(k)\,\mathrm{d}k=c\epsilon^{-(2/3)+\sigma}\int_0^\infty\frac{z^2}{(1+z^2)^{(11/6)-(\sigma/2)}}\exp(-\epsilon^2z^2)\,\mathrm{d}z \tag{13}$$

the exponent σ may also be interpreted as a measure of the infrared divergence of kinetic energy in the limit $\epsilon \to 0$ ($\sigma < 2/3$) [14–17, 20, 21].

The dynamic exponent z describes the dependence of the correlation time $\tau(k)$ upon k. For $z = \frac{2}{3}$, all 'eddies' have identical turnover time.

Now we are in a position to consider the limit $\epsilon \to 0$ in (9) and thereby to find the scaling law $\lambda(\epsilon)$ and an effective equation describing the long-time, large-distance behaviour of the one-point PDF of scalar field. However, before proceeding further we would like to make some comments concerning the asymptotic behaviour of (9) as $\epsilon \to 0$. One can expect that for a wide range of values of σ and z the diffusive scaling $\lambda(\epsilon) = \epsilon^2$ leads to the infrared divergence of (9) in the limit $\epsilon \to 0$. Hence one needs the renormalization procedure that makes the correlation term (9) finite. In what follows we consider only the cases in which the effect of infrared divergence plays a key role ($\sigma < 2/3$). One can anticipate that there are at least two very different asymptotic regimes when such a behaviour occurs giving the anomalous scaling laws [17]. One is obtained if we take the limit $\lambda/\epsilon^{(2/3)-z} \to 0$ when the effective correlation time $\lambda/\epsilon^{(2/3)-z}(1+k^2)^{-(1/3)+(z/2)}$ tends to zero. We refer to this regime as the 'fast turbulence limit' [17] and show below that the effective equation for $p^0(t, \boldsymbol{x}, \varphi) = \lim_{\epsilon\to 0} p^\epsilon(t, \boldsymbol{x}, \varphi)$ is a conventional diffusion-like equation while the scaling law is superdiffusive. The other regime can be obtained when $\lambda/\epsilon^{(2/3)-z} \to \infty$ and, therefore, the effective correlation time tends to infinity. We refer to this regime as the 'frozen turbulence limit' [17]. Unlike the first case no simple effective equation can be derived.

'Fast turbulence limit'. Consider the case in which

$$\frac{\lambda}{\epsilon^{(2/3)-z}} \to 0 \qquad \text{as } \epsilon \to 0. \tag{14}$$

The renormalization procedure must consist of determining such a scaling function $\lambda(\epsilon)$ so that the correlation term

$$Q^\epsilon = \frac{1}{\lambda^2\pi^2} \iiiint \exp\left[\mathrm{i}\frac{\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}{\epsilon} - \mathrm{i}\frac{\omega(t-\tau)}{\lambda}\right] E(k) \frac{\tau(k)}{1+(\tau(k)\omega)^2} \times k^{-2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{\partial^2}{\partial x_i \partial\varphi} \left\langle \frac{\delta\varphi(t/\lambda, \boldsymbol{x}/\epsilon)}{\delta v_j(\tau/\lambda, \boldsymbol{y}/\epsilon)} \delta(\varphi - \varphi^\epsilon) \right\rangle \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega\, \mathrm{d}\tau\, \mathrm{d}\boldsymbol{y} \tag{15}$$

has a non-trivial limit as $\epsilon \to 0$. The appropriate choice of $\lambda(\epsilon)$ is

$$\lambda(\epsilon) = \epsilon^{(2/3)+\sigma+z}.$$

It follows from (11), (14) and (15) that as ϵ tends to zero, the spectral density in the correlation term Q^ϵ loses its dependence on ω (after the frequency rescaling) and this implies that a scaling procedure generates a white-noise in time. This fact allows us to derive an equation for $p^0(t, \boldsymbol{x}, \varphi)$ in a closed form. It is important to note that since $\epsilon^2/\lambda \to 0$ the molecular diffusion is irrelevant. Therefore, the closure problem associated with the diffusion term is completely overcome in this limit. Because of the white-noise limit it is sufficient to calculate the one-time functional derivative

$$\frac{\delta\varphi(t/\lambda, \boldsymbol{x}/\epsilon)}{\delta v_j(t/\lambda, \boldsymbol{y}/\epsilon)} = -\epsilon \frac{\partial\varphi}{\partial x_i} \delta\left(\frac{\boldsymbol{x}}{\epsilon} - \frac{\boldsymbol{y}}{\epsilon}\right)$$

and hence

$$\lim_{\epsilon\to 0} Q^\epsilon = D_{\mathrm{T}}\left(\frac{\partial^2 p^0}{\partial x_1^2} + \frac{\partial^2 p^0}{\partial x_2^2} + \frac{\partial^2 p^0}{\partial x_3^2}\right) \tag{16}$$

provided that the integral

$$D_{\mathrm{T}} = \frac{4c}{3} \int_0^\infty \frac{k^2}{(1+k^2)^{(13/6)-(\sigma/2)-(z/2)}} \, \mathrm{d}k$$

determining the turbulent diffusion coefficient D_{T} converges, i.e. $\sigma + z < 4/3$. The requirement in (14) leads to the restriction $\sigma + 2z > 0$.

It follows from (8) and (16) that the effective equation describing the long-time, large-distance behaviour of the one-point PDF is given by

$$\frac{\partial p^0}{\partial t} = -\frac{\partial}{\partial \varphi}(\alpha_0 f(\varphi) p^0) + D_{\mathrm{T}} \left(\frac{\partial^2 p^0}{\partial x_1^2} + \frac{\partial^2 p^0}{\partial x_2^2} + \frac{\partial^2 p^0}{\partial x_3^2} \right) \tag{17}$$

provided

$$\sigma + 2z > 0 \qquad \sigma + z < 4/3 \qquad \sigma < 2/3$$

and $\alpha(\epsilon) = \alpha_0 \lambda(\epsilon)$.

The initial condition is $p^0(0, \boldsymbol{x}, \varphi) = \delta(\varphi - \varphi_0(\boldsymbol{x}))$.

The large-scale, long-time Lagrangian scaling for the particle displacement is superdiffusive, i.e.

$$x^2(t) \sim t^{6/(2+3\sigma+3z)}.$$

For the Kolmogorov–Obukhov turbulence $\sigma = 0, z = 0$, we recover the well known scaling $x^2(t) \sim t^3$ corresponding to the Richardson law [1, 17].

In summary, by using an exact renormalization theory and functional derivative technique, we are able without any *ad-hoc* appoximation to derive the effective equation for the long-time, large-distance form for the one-point probability density function of the scalar field advected by a three-dimensional random velocity field with arbitrary many spatial-temporal scales and infrared divergence in the limit of large Reynolds number. We show that when the spectral parameters of a random velocity field slightly deviate from their Kolmogorov–Obukhov values, the PDF equation for a scalar field can be determined exactly and, what is more, it is well described by a conventional diffusion theory, while the Lagrangian scaling function is found to be essentially superdiffusive.

Note that there are several possible directions to explore by the method developed here. First, one may study the higher-order statistics of a passive scalar including anomalous turbulent decay and large-scale intermittency [9, 10]. Also, one can extend the analysis to the turbulent Kolmogorov–Petrovskii–Piskunov dynamics of the reaction fronts in three-dimensional space [20–24].

The research was supported by EC project INTAS-94-2580.

References

[1] Monin A S and Yaglom A M 1987 *Statistical Fluid Dynamics* (Cambridge, MA: MIT Press)
[2] Foster D, Nelson D and Stefen M 1977 *Phys. Rev.* A **16** 732
[3] Rose H A 1977 *J. Fluid Mech.* **81** 719
[4] Kraichnan R 1987 *Complex Syst.* **1** 805
[5] Yakhot V and Orszag S 1986 *J. Sci. Comput.* **1** 3
[6] Sinai Y G and Yakhot V 1989 *Phys. Rev. Lett.* **63** 1962
[7] Chen Y, Chen S and Kraichnan R H 1989 *Phys. Rev. Lett.* **63** 2657
[8] Pumir A, Shraiman B I and Siggia E D 1991 *Phys. Rev. Lett.* **66** 2984
[9] Shraiman B I and Siggia E D 1994 *Phys. Rev.* **49** 2912
[10] Majda A 1993 *Phys. Fluids* A **5** 1963

[11] Majda A 1994 *J. Stat. Phys.* **75** 1153
[12] Chertkov M, Falkovich G and Lebedev V 1996 *Phys. Rev. Lett.* **76** 3707
[13] Pope S B 1986 *Prog. Eng. Comb. Sci.* **11** 119
[14] Avellaneda M and Majda A 1990 *Comm. Math. Phys.* **131** 381
[15] Avellaneda M and Majda A 1992 *Comm. Math. Phys.* **146** 139
[16] Avellaneda M and Majda A 1992 *Phys. Fluids* A **4** 41
[17] Avellaneda M and Majda A 1992 *Phys. Rev. Lett.* **68** 3028
[18] Klyatskin V I and Tatarskii V I 1974 *Sov. Phys.–Usp.* **16** 494
[19] Moss F and McClintock P V E (eds) 1989 *Noise in Nonlinear Dynamical Systems* vol 1 (Cambridge: Cambridge University Press)
[20] Fedotov S P 1995 *Phys. Rev.* E **52** 3835
[21] Fedotov S P 1997 *Phys. Rev.* E **55** N3
[22] Souganidis P and Majda A 1994 *Nonlinearity* **7** 1
[23] Fedotov S P 1995 *J. Phys. A: Math. Gen.* **28** L461
[24] Fedotov S P 1996 *J. Phys. A: Math. Gen.* **29** L517